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New generalized Verma modules and multilinear intertwining differential operators

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Abstract

This paper contains *two* interrelated developments. *First* are proposed new generalized Verma modules. They are called k-Verma modules, $k \in \mathbb{N}$, and coincide with the usual Verma modules for $k = 1$. As a vector space a k-Verma module is isomorphic to the symmetric tensor product of k copies of the universal enveloping algebra $U(\mathcal{G}^-)$, where $\mathcal{G}^-$ is the subalgebra of lowering generators in the standard triangular decomposition of simple Lie algebra $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$. The *second* development is the proposal of a procedure for the construction of *multilinear* intertwining differential operators for semisimple Lie groups G. This procedure uses k-Verma modules and coincides for $k = 1$ with a procedure for the construction of *linear* intertwining differential operators. For all k a central role is played by the singular vectors of the k-Verma modules. Explicit formulae for series of such singular vectors are given. Using these are given explicitly many new examples of multilinear intertwining differential operators. In particular, for $G = SL(2, \mathbb{R})$ are given explicitly all bilinear intertwining differential operators. Using the latter, as an application are constructed $\frac{1}{2}n$-differentials for all $n \in 2\mathbb{N}$, the ordinary Schwarzian being the case $n = 4$. © 1998 Elsevier Science B.V.

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1. Introduction

1.1

Operators intertwining representations of Lie groups play a very important role both in mathematics and physics. To recall the notions, consider a Lie group G and two representations T, T' of G acting in the representation spaces C, C', which may be Hilbert, Fréchet, etc. An *intertwining operator* $\mathcal{I}$ for these two representations is a continuous linear map

$$\mathcal{I} : C \to C', \quad \mathcal{I} : f \mapsto j, \qquad f \in C, \;\; j \in C', \tag{1.1}$$

such that

$$\mathcal{I} \circ T(g) = T'(g) \circ \mathcal{I} \quad \forall g \in G. \tag{1.2}$$

An important application of the intertwining operators is that they produce canonically invariant equations. Indeed, in the setting above the equation

$$\mathcal{I} f = j \tag{1.3}$$

is a G-invariant equation. These are very useful in the applications, recall, e.g., the well-known examples of Dirac, Maxwell equations. The intertwining operators are also very relevant for analysing the structure of representations of Lie groups, especially of semisimple (or reductive) Lie groups, cf., e.g., [22,24,31,32]. There are two types of intertwining operators: integral and differential. For the integral intertwining operators, which we shall not discuss here, we refer to [22,31] for the mathematical side and to [11] for explicit examples and applications. For the intertwining differential operators we refer to [10,24,32] (for early examples and partial cases see, e.g., [1,2,5,6,9,11,12,14,16–18,26,27,30]).

1.2

In the present paper we discuss *multilinear intertwining differential operators* such that

$${}_k\mathcal{I} : \underbrace{f \otimes \cdots \otimes f}_{k} \mapsto j, \quad f \in C, \;\; j \in C', \tag{1.4}$$

$${}_k\mathcal{I} \circ \underbrace{T(g) \otimes \cdots \otimes T(g)}_{k} = T'(g) \circ {}_k\mathcal{I} \quad \forall g \in G. \tag{1.5}$$

Clearly, for $k = 1$ (1.4) and (1.5) reduce to (1.1) and (1.2), respectively.

Let us give an example of such an operator for $k = 2$. Let $G = SL(2, \mathbb{R})$ and consider C^∞-functions so that the representation is acting as [15]:

$$T^c(g) f(x) = |\delta - \beta x|^{-c} f\left(\frac{\alpha x - \gamma}{\delta - \beta x}\right), \quad \delta - \beta x \neq 0, \tag{1.6a}$$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1, \;\; \alpha, \beta, \gamma, d \in \mathbb{R}, \tag{1.6b}$$

where $c \in \mathbb{C}$ is a parameter characterizing the representation (for more details see Section 4). Consider now the operator

$$ {}_2\mathcal{I}(f) = f'''f' - \tfrac{3}{2}(f'')^2, \tag{1.7} $$

where f', f'', f''' are the first, second, third, respectively, derivatives of f. Let us denote the space of functions with (1.6) as transformation rule by C^c. Then it is easy to show that ${}_2\mathcal{I}$ has the following intertwining property:

$$ {}_2\mathcal{I} : f \otimes f \mapsto j, \quad f \in C^0, \ j \in C^8, \tag{1.8} $$

$$ {}_2\mathcal{I} \circ (T^0(g) \otimes T^0(g)) = T^8(g) \circ {}_2\mathcal{I} \quad \forall g \in G. \tag{1.9} $$

1.3

We would like to note that our problem is related to the problem of finding invariant n-differentials. In the simple example above such a relation is straightforward. Indeed an example of invariant quadratic differential is the Schwarzian (cf., e.g., [20]):

$$ \begin{aligned} \mathrm{Sch}\ (f) &\doteq \left(\frac{f'''}{f'} - \frac{3}{2}\frac{(f'')^2}{(f')2} \right) (\mathrm{d}x)^2, \\ \mathrm{Sch}\ (f \circ f_0) &= \mathrm{Sch}(f(x)) \circ f_0, \quad f_0(x) = \frac{\alpha x - \gamma}{\delta - \beta x}, \\ \mathrm{Sch}\ (f_0) &= 0. \end{aligned} \tag{1.10} $$

Of course, such direct relations are an artifact of the simplicity of the situation. Our setting is more general than the problem of finding invariants as in (1.10) since it allows in principle arbitrary representation parameters. Below we give such operators for any semisimple Lie group.

For other examples of multilinear invariant operators see, e.g., [3,13]. These examples rely on adaptations of the classical polynomial invariant theory of Weyl [29]. Another approach is to use invariant differentiation with respect to a Cartan connection [7].

1.4

Our approach is different from those of [3,7,13], mentioned above. It is a natural generalization of the $k = 1$ procedure of [10]. More than that, the present paper contains *two* interrelated developments. *First* we propose *new generalized Verma modules*. They are called k-Verma modules, $k \in \mathbb{N}$, and coincide with the usual Verma modules for $k = 1$. As a vector space a k-Verma module is isomorphic to the symmetric tensor product of k copies of the universal enveloping algebra $U(\mathcal{G}^-)$, where $\mathcal{G}^-$ is the subalgebra of lowering generators in the standard triangular decomposition of a simple Lie algebra $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$. The *second* development is the proposal of a procedure for the construction of *multilinear intertwining differential operators* for semisimples Lie groups G. This procedure uses the k-Verma modules and coincides for $k = 1$ with our procedure for the construction of *linear* intertwining differential operators [10]. For all k central role is played by the singular vectors of the k-Verma modules. Explicit formulae for series of such singular vectors are given for arbitrary $\mathcal{G}$. Using these are given explicitly many new examples of multilinear intertwining differential operators. In particular, for $G = SL(2, \mathbb{R})$ are given explicitly all bilinear intertwining differential operators. Using the latter, as an application are constructed $\frac{1}{2}n$-differentials for all $n \in 2\mathbb{N}$, the ordinary Schwarzian being the case $n = 4$.

1.5

The organization of the paper is as follows.

In Section 2 we first recall the usual Verma modules formulating their reducibility conditions in a way suitable for our purposes. Then we introduce the new generalization of the Verma modules, which we call k-Verma modules, and we give some of their general properties.

In Section 3 we consider the singular vectors of the k-Verma modules. Using the singular vectors we show that k-Verma modules are always reducible independently of the highest weight in sharp contrast with the ordinary Verma modules ($k = 1$). We also give many important explicit examples of singular vectors for $k = 2, 3$. For bi-Verma (= 2-Verma) modules we give the general explicit formula for a class of singular vectors, which exhausts all possible cases for $\mathcal{G} = sl(2)$.

In Section 4 we first recall the procedure of [10] for the construction of linear intertwining differential operators. Then we generalize this procedure for the construction of multilinear intertwining differential operators. This is a general result which produces a multilinear intertwining differential operator for every singular vector of a k-Verma module, the procedure of [10] being obtianed for $k = 1$.

In Section 5 we study bilinear operators for $G = SL(n, \mathbb{R})$ mentioning also which results are extendable to $SL(N, \mathbb{C})$. We give explicit formulae for all bilinear intertwining differential operators for $\mathcal{G} = sl(2, \mathbb{R})$ and $SL(2, \mathbb{R})$, noting the difference between the algebra and group invariants. We study in some detail partial cases, in particular, an infinite hierarchy of even order intertwining differential operators producing $\frac{1}{2}n$-differentials for all $n \in 2\mathbb{N}$, the ordinary Schwarzian being the case $n = 4$. We also give many examples for $SL(2, \mathbb{R})$.

In Section 6 we give some examples which illustrate additional new features of the multilinear intertwining differential operators for $k > 2$.

In Appendix A we have summarized the notions of tensor, symmetric and universal enveloping algebras.

2. k-Verma modules

2.1

Let $F = \mathbb{C}$ or $F = \mathbb{R}$. Let $\mathcal{G}$ be a semisimple Lie algebra over $F = \mathbb{C}$ or a split real semisimple Lie algebra over $F = \mathbb{R}$. Thus $\mathcal{G}$ has a triangular decomposition: $\mathcal{G} =$

$\mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^-$, where $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$ (the maximally non-compact Cartan subalgebra for $F = \mathbb{R}$), $\mathcal{G}^+$, respectively $\mathcal{G}^-$, are the positive, respectively, negative root vector spaces of the root system $\Delta = \Delta(\mathcal{G}, \mathcal{H})$, corresponding to the decomposition $\Delta = \Delta^+ \cup \Delta^-$ into positive and negative roots. (For $F = \mathbb{R}$ this decomposition is a partial case of a Bruhat decomposition.) In particular, one has: $\mathcal{G}^\pm = \oplus_{\beta \in \Delta^+} \mathcal{G}^\pm_\beta$. In our cases $\dim \mathcal{G}^p_\beta m = 1, \forall \beta \in \Delta^+$, and further $X^\pm_\beta$ will denote a vector spanning $\mathcal{G}^\pm_\beta$. Let Δ_S be the system of simple roots of Δ. Let $\Gamma^+ \in \mathcal{H}^*$ denote the set of dominant weights, i.e., $\nu \in \Gamma^+$ iff $(\nu, \alpha_i^\vee) \in \mathbb{N}_+$ for all $\alpha_i \in \Delta_S$. Let $U(\mathcal{G})$ be the universal enveloping algebra of $\mathcal{G}$ with unit vector denoted by 1_u. (The notions of tensor, symmetric and universal enveloping algebras are recalled in Appendix A.)

Let us recall that a *Verma module* V^Λ is defined as the HWM over $\mathcal{G}$ with highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V^\Lambda$, induced from the one-dimensional representation $V_0 \cong \mathbb{C}v_0$ of $U(\mathcal{B})$, where $\mathcal{B} = \mathcal{B}^+ = \mathcal{H} \oplus \mathcal{G}^+$ is a Borel subalgebra of $\mathcal{G}$, such that

$$\begin{aligned} &X v_0 = 0, \quad X \in \mathcal{G}^+, \\ &H v_0 = \Lambda(H) v_0, \quad H \in \mathcal{H}. \end{aligned} \tag{2.1}$$

Thus one has

$$V^\Lambda \cong U(\mathcal{G}) \otimes_{U(\mathcal{B})} v_0 \cong U(\mathcal{G}^-) \otimes v_0 \tag{2.2}$$

(isomorphisms between vector spaces). One considers V^Λ as a left $U(\mathcal{G})$-module (w.r.t. multiplication in $U(\mathcal{G})$).

Verma modules are generically irreducible. A Verma module is reducible iff it has singular vectors (one or more) [4]. A *singular vector* of a Verma module V^Λ is a vector $v_s \in V^\Lambda$, such that $v_s \notin F 1_u \otimes v_0$ and

$$\begin{aligned} &X v_s = 0, \quad X \in \mathcal{G}^+, \\ &H v_s = (\Lambda(H)) - \mu(H) v_s, \quad \mu \in \Gamma^+, \ \mu \neq 0, \ H \in \mathcal{H}. \end{aligned} \tag{2.3}$$

The space $U(\mathcal{G}^-) \otimes v_s$ is a submodule of V^Λ isomorphic to the Verma module $V^{\Lambda-\mu} = U(\mathcal{G}^-) \otimes v'_0$ where v'_0 is the highest weight vector of $V^{\Lambda-\mu}$; the isomorphism being realized by $v_s \mapsto 1_u \otimes v'_0$. Furthermore, there exists (at least one) decomposition $\mu = \sum_{i=1}^n m_i \beta_i, m_i \in \mathbb{N}, \beta_i \in \Delta^+$; the latter statement in the case $n = 1$ means that $\mu = m\beta, m \in \mathbb{N}, \beta \in \Delta^+$. For each such decomposition there exists a composition of embeddings of the Verma modules $V_i \equiv V^{\Lambda - m_i\beta_i}$ which thus form a nested sequence of submodules, so that V_i is a submodule of $V_{i-1}, i = 1, \ldots, n$, $V_0 \equiv V^\Lambda$. Each such submodule is generated by a singular vector of weight $m_i\beta_i$. The singular vector of weight $m\beta$ is given by [10]:

$$v_s = v_s^{m\beta} = \mathcal{P}^{m\beta}(X_1^-, \ldots, X_l^-) \otimes v_0, \tag{2.4}$$

where $\mathcal{P}^{m\beta}$ is a homogeneous polynomial in its variables of degrees mn_i, where $n_i \in \mathbb{Z}_+$ come from $\beta = \sum n_i \alpha_i$, α_i form the system of simple roots Δ_S. The polynomial $\mathcal{P}^\beta_m$ is unique up to a non-zero multiplicative constant. From this follows that the singular vector of weight μ is given by

$$v_s = v_s^{\mu} = \mathcal{P}^{m_n\beta_n} \dots \mathcal{P}^{m_1\beta_1} \otimes v_0. \tag{2.5}$$

Finally, we should mention that in this setting the highest weight satisfies

$$(\Lambda + \rho, \beta_1^{\vee}) - m_1 = (\Lambda + \rho)(H_{\beta_1}) - m_1 = 0, \tag{2.6}$$

where ρ is half the sum of all positive roots, $\alpha^{\vee} \equiv 2\alpha/(\alpha,\alpha)$ for any $\alpha \in \Delta$, $(\ ,\ )$ is the scalar product in $\mathcal{H}^*$, $H_\alpha \in \mathcal{H}$ corresponds to $\alpha \in \Delta^+$ under the isomorphism $\mathcal{H} \cong \mathcal{H}^*$. As a consequence one has

$$\begin{aligned} &(\Lambda_{i-1} + \rho, \beta_i^{\vee}) = m_i, \quad i = 1, \dots, n, \\ &\Lambda_0 = \Lambda, \qquad \Lambda_i = \Lambda_{i-1} - m_i\beta_i. \end{aligned} \tag{2.7}$$

One should note that condition (2.6) is in fact necessary and sufficient for the reducibility of a Verma module. Thus one may say equivalently that the Verma module V^Λ is reducible iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ so that holds [4]:

$$(\Lambda + \rho, \beta^{\vee}) - m = (\Lambda + \rho)(H_\beta) - m = 0. \tag{2.8}$$

We have chosen a different exposition here since in the generalization of the Verma modules we introduce below we do not rely on an analogue of (2.8) and reducibility is discussed via singular vectors.

2.2

We introduce now a generalization of the Verma modules. Let k be a natural number, let $\mathcal{T}_k(\mathcal{G})$ be the tensor product:

$$\mathcal{T}_k(\mathcal{G}) \doteq T_k(U(\mathcal{G})) = \underbrace{U(\mathcal{G}) \otimes \cdots \otimes U(\mathcal{G})}_{k}, \tag{2.9}$$

and let $\mathcal{S}_k(\mathcal{G})$ be the symmetric tensor product

$$\mathcal{S}_k(\mathcal{G}) \doteq S_k(U(\mathcal{G})) = T_k(U(\mathcal{G}))/I_k(U(\mathcal{G})). \tag{2.10}$$

Then arbitrary elements of $\mathcal{S}_k(\mathcal{G})$ shall be denoted as follows:

$$u = \{u_1 \otimes \cdots \otimes u_k\}, \quad u_j \in U(\mathcal{G}), \tag{2.11}$$

where $\{\cdots\}$ denotes the symmetric tensor product which is preserved under abitrary permutations $u_i \longleftrightarrow u_j$.

Definition 1. A *k-Verma module* ${}_kV^\Lambda$ is a highest weight module over $\mathcal{G}$ induced from the one-dimensional representation of $\mathcal{B}$ (*cf.* (2.1)) so that

$${}_kV^\Lambda \cong \mathcal{S}_k(\mathcal{G})\hat{\otimes} v_0 \cong \mathcal{S}_k(\mathcal{G}^-)\hat{\otimes} v_0, \quad \hat{\otimes} \equiv \otimes_{U(\mathcal{B})} \tag{2.12}$$

(isomorphisms between vector spaces). ${}_kV^\Lambda$ is considered a left $U(\mathcal{G})$-module (w.r.t. multiplication in $U(\mathcal{G})$). Denoting arbitrary v of ${}_kV^\Lambda$ consistently with (2.11):

$$v = \{u_1 \otimes \cdots \otimes u_k\} \hat{\otimes} v_0, \quad u_j \in U(\mathcal{G}^-), \tag{2.13}$$

we define the action of $U(\mathcal{G})$ as follows:

$$Xv = \sum_{j=1}^{k} \{u_1 \otimes \cdots \otimes Xu_j \otimes \cdots \otimes u_k\} \hat{\otimes} v_0, \quad X \in U(\mathcal{G}). \tag{2.14}$$

Remark 1. Clearly, 1-Verma modules are usual Verma modules.

Corollary 1 (from Definition 1). *Let $H \in \mathcal{H}$, let u_j be from the PBW basis of $U(\mathcal{G}^-)$, let μ_j be the (negative) weight of u_j, i.e., $[H, u_j] = -\mu_j(H)u_j$. Then we have*

$$\begin{aligned} Hv &= \sum_{j=1}^{k} \{u_1 \otimes \cdots \otimes Hu_j \otimes \cdots \otimes u_k\} \hat{\otimes} v_0 \\ &= \sum_{j=1}^{k} \{u_1 \otimes \cdots \otimes u_j \otimes \cdots \otimes u_k\} \hat{\otimes} (H - \mu_j(H)) v_0 \\ &= \left(k\Lambda(H) - \sum_{j=1}^{k} \mu_j(H) \right) v. \end{aligned} \tag{2.15}$$

2.3

We need some more notation to proceed further. Let $P(\mu)$ be the number of ways $\mu \in \Gamma^+$ can be presented as a sum of positive roots β. (In general, each root should be taken with its multiplicity; however, in the cases here all multiplicities are equal to 1.) By convention $P(0) \equiv 1$. Let ${}_k\Gamma^+ \doteq \Gamma^+ \times \cdots \times \Gamma^+$, k factors. Let ${}_k\mu = (\mu_1, \ldots, \mu_k) \in {}_k\Gamma^+$, $\mu_j \in \Gamma^+$, and let $\sigma({}_k\mu) \doteq \sum_{j=1}^{k} \mu_j \in \Gamma^+$. Let $\mu \in \Gamma^+$ and $P_k(\mu)$ be the number of elements ${}_k\nu = (\nu_1, \ldots, \nu_k) \in {}_k\Gamma^+$ such that $\sigma({}_k\nu) = \mu$, each such element being taken with multiplicity $\prod_{j=1}^{k} P(\nu_j)$. Clearly, $P_k(0) = 1$, $P_k(\beta) = k$, $\forall \beta \in \Delta_S$. Finally we define

$${}_k\Gamma^+_> \doteq \{{}_k\mu = (\mu_1, \ldots, \mu_k) \in {}_k\Gamma^+ \mid \mu_1 \geq \cdots \geq \mu_k\}, \tag{2.16}$$

where some ordering of $\mathcal{H}^*$ (e.g., the lexicographical one) is implemented. Let $\mu \in \Gamma^+$ and let $P_k^>(\mu)$ be the number of elements ${}_k\nu = (\nu_1, \ldots, \nu_k) \in {}_k\Gamma^+_>$ such that $\sigma({}_k\nu) = \mu$, each such elements being taken with multiplicity $\prod_{j=1}^{k} P(\nu_j)$. Clearly, $P_k(0) = 1$, $P_k^>(\beta) = 1$ $\forall \beta \in \Delta_S$. Now one can prove the following:

Proposition 1. *Let $\Lambda \in \mathcal{H}^*$ and let*

$${}_kV^\Lambda_\mu \doteq \{v \in {}_kV^\Lambda \mid Hv = (k\Lambda(H) - \mu(H))v\}. \tag{2.17}$$

Then we have

$$ {}_kV^\Lambda = \bigoplus_{\mu\in\Gamma^+} {}_kV^\Lambda_\mu, \tag{2.18a}$$

$$ \dim {}_kV^\Lambda_\mu = P^>_k(\mu), \tag{2.18b}$$

$$ {}_kV^\Lambda_\mu = \sum_{\substack{{}_k\nu\in{}_k\Gamma^+_>\\ \sigma({}_k\nu)=\mu}} \sum_{\substack{\beta^1_i\in\Delta^+\\ \sum\beta^1_i=\nu_1\\ \beta^1_1\leq\beta^1_2\leq\cdots\leq\beta^1_{n_1}}} \cdots \sum_{\substack{\beta^k_i\in\Delta^+\\ \sum\beta^k_i=\nu_k\\ \beta^k_1\leq\beta^k_2\leq\cdots\leq\beta_{n^k_k}}} \times \{X^-_{\beta^1_1}\cdots X^-_{\beta^1_{n_1}}\otimes\cdots\otimes X^-_{\beta^k_1}\cdots X^-_{\beta^k_{n_k}}\}\hat{\otimes}Fv_0, \tag{2.18c}$$

$$ {}_kV^\Lambda_0 = \{1_\mathrm{u}\otimes\cdots\otimes 1_\mathrm{u}\}\hat{\otimes}Fv_0, \tag{2.18d}$$

$$ {}_kV^\Lambda = \mathcal{S}_k(\mathcal{G}^-)_k V^\Lambda_0, \qquad \mathcal{G}^+{}_kV^\Lambda_0 = 0, \tag{2.18e}$$

where in (2.18c) *the ordering of the root system inherited from the ordering of* $\mathcal{H}^*$ *is implemented.*

Proof. Completely analogous to the classical case $k = 1$ [8]. □

3. Singular vectors of k-Verma modules

3.1

In contrast to the ordinary Verma modules ($k = 1$), the k-Verma modules for $k \geq 2$ are reducible independently of the highest weight, which is natural taking into account their tensor product character. This we show by exhibiting singular vectors for arbitrary highest weights.

We call a *singular vector* of a k-Verma module ${}_kV^\Lambda$ a vector $v_\mathrm{s} \in {}_kV^\Lambda_0$ such that $v_\mathrm{s} \notin {}_kV^\Lambda_0$ and

$$ Xv_\mathrm{s} = 0, \quad X \in \mathcal{G}^+, \tag{3.1a}$$

$$ Hv_\mathrm{s} = (k\Lambda(H) - \mu(H))v_\mathrm{s}, \quad \mu \in \Gamma^+, \ \mu \neq 0, \ H \in \mathcal{H}, \tag{3.1b}$$

i.e., v_s is homogeneous: $v_\mathrm{s} \in {}_kV^\Lambda_\mu$ for some $\mu \in \Gamma^+$. For $k = 1$ (3.1) coincide with (2.3).

The space $\mathcal{S}_k(\mathcal{G}^-)v_\mathrm{s}$ is a submodule of ${}_kV^\Lambda$ isomorphic to the Verma module ${}_kV^{k\Lambda-\mu} = \mathcal{S}_k(\mathcal{G}^-)\otimes v'_0$, where v'_0 is the highest weight vector of ${}_kV^{k\Lambda-\mu}$; the isomorphism being realized by $v_\mathrm{s} \mapsto \{1_\mathrm{u}\otimes\cdots\otimes 1_\mathrm{u}\}\hat{\otimes}v'_0$.

In the next two sections we show some explicit examples for the cases $k = 2, 3$.

3.2

We consider now the case $k = 2$, i.e., *bi-Verma* (= 2-Verma) modules. We take a weight $\mu = n\alpha$, where $n \in \mathbb{N}$ and $\alpha \in \Delta_\mathrm{S}$ is any root. We have $\dim {}_2V^\Lambda_{n\alpha} = [n/2]+1$, where $[x]$ is the largest integer not exceeding x. The possible singular vectors have the following form:

$$ {}_2v_s^{n\alpha} = \sum_{j=0}^{[n/2]} \gamma_{nj}^{\Lambda}\{(X_\alpha^-)^{n-j} \otimes (X_\alpha^-)^j\}\hat{\otimes} v_0. \tag{3.2} $$

The coefficients γ_{nj}^{Λ} are determined form condition (3.1a) with $X = X_\alpha^+$ – all other cases of (3.1) are fulfilled automatically. Thus we have:

Proposition 2. *The singular vectors of the bi-Verma (= 2-Verma) module ${}_2V^{\Lambda}$ of weight $\mu = n\alpha$, where $n \in \mathbb{N}$ and $\alpha \in \Delta_S$ is any simple root, are given by formula* (3.2) *with the coefficients γ_{nj} given explicitly by*

$$ \begin{aligned} \gamma_{nj}^{\Lambda} &= \gamma_0 \gamma(n, \Lambda(H))(-1)^j (2 - \delta_{j,n/2}) \\ &\quad \times \binom{n}{j} \frac{\Gamma(\Lambda(H) + 1 - n + j)\Gamma(\Lambda(H) + 1 - j)}{\Gamma(\Lambda(H) + 1 - n)\Gamma(\Lambda(H) + 1 - [n/2])}, \\ \gamma(n, \Lambda(H)) &= \begin{cases} 1 & \text{for } n \text{ even and arbitrary } \Lambda(H), \\ 1 & \text{for } n \text{ odd}, \\ & \quad \Lambda(H) = n-1, n-2, \ldots, \frac{1}{2}(n-1), \\ 0 & \text{for } n \text{ odd}, \\ & \quad \Lambda(H) \neq n-1, n-2, \ldots, \frac{1}{2}(n-1), \end{cases} \end{aligned} \tag{3.3} $$

and γ_0 is an arbitrary non-zero constant.

Proof. Follows from the direct verification. □

We give the lowest cases of the above general formula for illustration (fixing the overall constant γ_0 appropriately):

$$ {}_2v_s^{\alpha} = \{X_\alpha^- \otimes 1_u\}\hat{\otimes} v_0, \quad \Lambda(H) = 0, \tag{3.4a} $$

$$ \begin{aligned} {}_2v_s^{2\alpha} &= \{\Lambda(H)(X_\alpha^-)^2 \otimes 1_u \\ &\quad - (\Lambda(H) - 1)X_\alpha^- \otimes X_\alpha^-\}\hat{\otimes} v_0 \quad \forall \Lambda(H), \end{aligned} \tag{3.4b} $$

$$ \begin{aligned} {}_2v_s^{3\alpha} &= \{\Lambda(H)(X_\alpha^-)^3 \otimes 1_u \\ &\quad - 3(\Lambda(H) - 2)(X_\alpha^-)^2 \otimes X_\alpha^-\}\hat{\otimes} v_0, \quad \Lambda(H) = 1, 2, \end{aligned} \tag{3.4c} $$

$$ \begin{aligned} {}_2v_s^{4\alpha} &= \{\Lambda(H)(\Lambda(H) - 1)(X_\alpha^-)^4 \otimes 1_u \\ &\quad - 4(\Lambda(H) - 1)(\Lambda(H) - 3)(X_\alpha^-)^3 \otimes X_\alpha^- \\ &\quad + 3(\Lambda(H) - 2)(\Lambda(H) - 3)(X_\alpha^-)^2 \otimes (X_\alpha^-)^2\}\hat{\otimes} v_0 \quad \forall \Lambda(H). \end{aligned} \tag{3.4d} $$

Proposition 2 confirms that bi-Verma modules are always reducible since they possess singular vectors independently of Λ. In fact, they have an infinite number of singular vectors of weights $n\alpha_i$, for any even positive integer n and any simple root α_i. Moreover, they possess singular vectors of other weights, also independent of Λ. For example we consider weights $\mu_n = n\beta = n(\alpha_1 + \alpha_2)$, where β is a positive root, and α_1, α_2 are two simple roots, e.g., of equal minimal length (for simplicity). Then there exist singular vectors of these weights given by, e.g.,

$$\begin{aligned}
{}_2v_s^{\beta} &= \{\Lambda_1 X_1^- X_2^- \otimes 1_u - \Lambda_2 X_2^- X_1^- \otimes 1_u \\
&\quad - (\Lambda_1 + \Lambda_2 + 1) X_1^- \otimes X_2^- \} \hat{\otimes} v_0 \quad \forall \Lambda, \qquad (3.5)\\
\Lambda_a &\equiv \Lambda(H_a), \quad a = 1, 2,\\
{}_2v_s^{2\beta} &= a_1 (X_3^-)^2 \otimes 1_u + a_2 X_2^- X_3^- X_1^- \otimes 1_u \\
&\quad + a_3 (X_2^-)^2 (X_1^-)^2 \otimes 1_u \\
&\quad + b_1 X_3^- X_1^- \otimes X_2^- + b_2 X_2^- X_3^- \otimes X_1^- \\
&\quad + c_1 X_2^- (X_1^-)^2 \otimes X_2^- + c_2 (X_2^-)^2 X_1^- \otimes X_1^- \\
&\quad + d_1 X_3^- \otimes X_3^- + d_2 X_3^- \otimes X_2^- X_1^- \\
&\quad + d_3 X_2^- X_1^- \otimes X_2^- X_1^- + d_4 (X_2^-)^2 \otimes (X_1^-)^2 \quad \forall \Lambda, \qquad (3.6a)
\end{aligned}$$

where for the two solutions (present in this case) the coefficients are:

$$\begin{aligned}
a_1 &= \Lambda_1^2 \Lambda_2 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_2 + 1),\\
a_2 &= -\Lambda_1 \Lambda_2 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 - \Lambda_2 - 2),\\
a_3 &= -\Lambda_1 \Lambda_2 (\Lambda_1 + \Lambda_2 + 1),\\
b_1 &= -\Lambda_1 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2)(\Lambda_2 + 2),\\
b_2 &= \Lambda_1 \Lambda_2 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2),\\
c_1 &= \Lambda_1 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2), \qquad (3.6b)\\
c_2 &= \Lambda_2 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2),\\
d_1 &= -\Lambda_1^2 (\Lambda_1 + \Lambda_2)(\Lambda_2^2 + \Lambda_2 + 1),\\
d_2 &= \Lambda_1 (\Lambda_1 + \Lambda_2)(\Lambda_1 \Lambda_2 + 2\Lambda_1 - \Lambda_2^2 + \Lambda_2),\\
d_3 &= -(\Lambda_1 + \Lambda_2)(\Lambda_1^2 + \Lambda_1 \Lambda_2 + \Lambda_2^2),\\
d_4 &= 0,
\end{aligned}$$

$$\begin{aligned}
a_1 &= 2\Lambda_1^2 \Lambda_2 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_2^2 - \Lambda_2 - 1 + \Lambda_1 \Lambda_2),\\
a_2 &= -2\Lambda_1 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2 - 1)(\Lambda_1 - \Lambda_2)(\Lambda_2 + 1),\\
a_3 &= \Lambda_1 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2 - 1)(\Lambda_1 - \Lambda_2),\\
b_1 &= -2\Lambda_1 (\Lambda_1 + \Lambda_2 + 1)^2 (\Lambda_1 + \Lambda_2)(\Lambda_2 - 1),\\
b_2 &= 2\Lambda_1 (\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2)(\Lambda_2^2 - 2\Lambda_2 - 1 + \Lambda_1 \Lambda_2 + \Lambda_1),\\
c_1 &= 0, \qquad (3.6c)\\
c_2 &= -2(\Lambda_1 + \Lambda_2 + 1)(\Lambda_1 + \Lambda_2 - 1)(\Lambda_1 + \Lambda_2)(\Lambda_1 - \Lambda_2),\\
d_1 &= -2\Lambda_1^2 (\Lambda_1 + \Lambda_2)(\Lambda_2 - 1)(\Lambda_1 \Lambda_2 + \Lambda_1 + \Lambda_2^2),\\
d_2 &= -2\Lambda_1 (\Lambda_1 + \Lambda_2)(\Lambda_2 - 1)(\Lambda_2^2 - 2\Lambda_2 - 1 - \Lambda_1^2),\\
d_3 &= -2(\Lambda_1 + \Lambda_2)(\Lambda_2 - 1)(\Lambda_1 \Lambda_2 + \Lambda_1 + \Lambda_2^2),\\
d_4 &= \Lambda_1 (\Lambda_1 + \Lambda_2)(\Lambda_1 + \Lambda_2 + 1)^2.
\end{aligned}$$

Furthermore, the structure of the Verma modules is additionally complicated since there is no factorization of the singular vectors as for $k = 1$, cf. (2.5). For the latter consider the weight $\mu_0 = 2\alpha_1 + \alpha_2$, with α_1, α_2 as in the previous example. Then there exists a singular vector of this weight given by

$$
\begin{aligned}
{}_2v_s^{\mu_0} = \{&(2\Lambda_1\Lambda_2 + \Lambda_1 - \Lambda_2 - 1)(X_1^-)^2 X_2^- \otimes 1_u \\
&- 2\Lambda_1\Lambda_2 X_1^- X_2^- X_1^- \otimes 1_u \\
&+ (\Lambda_1 + \Lambda_2 + 1)(X_1^-)^2 \otimes X_2^- \\
&- 2(\Lambda_1 - 1)(\Lambda_2 + 1) X_1^- X_2^- \otimes X_1^- \\
&+ 2\Lambda_2(\Lambda_1 - 1) X_2^- X_1^- \otimes X_1^-\}\hat{\otimes} v_0 \quad \forall \Lambda.
\end{aligned} \tag{3.7}
$$

3.3

Here we consider the case $k = 3$, i.e., *tri-Verma* (=3-Verma) modules over arbitrary $\mathcal{G}$. Consider a weight $\mu = n\alpha$, where $n \in \mathbb{N}$ and $\alpha \in \Delta_S$ is any simple root. We first note the dimension of the weight space

$$
\dim {}_3V^\Lambda_{n\alpha} = (n - 3[n/6])(1 + [n/6]) + \delta_{n/6,[n/6]}. \tag{3.8}
$$

The possible singular vectors have the following form:

$$
{}_3v_s^{n\alpha} = \sum_{\substack{j,k\in\mathbb{Z}_+ \\ n-j-k\ge j\ge k}} \gamma^\Lambda_{njk}\{(X_\alpha^-)^{n-j-k} \otimes (X_\alpha^-)^j \otimes (X_\alpha^-)^k\}\hat{\otimes} v_0. \tag{3.9}
$$

The coefficients γ^Λ_{njk} are determined from condition (3.1a) with $X = X_\alpha^+$ – all other cases in (3.1) are fulfilled automatically. We give now the singular vectors for $n \le 6$ denoting $\hat{\Lambda} \equiv \Lambda(H)$:

$$
{}_3v_s^{\alpha} = \{X_\alpha^- \otimes 1_u \otimes 1_u\}\hat{\otimes} v_0, \quad \hat{\Lambda} = 0, \tag{3.10a}
$$

$$
\begin{aligned}
{}_3v_s^{2\alpha} = \{&\hat{\Lambda}(X_\alpha^-)^2 \otimes 1_u \hat{\otimes} 1_u \\
&- (\hat{\Lambda} - 1) X_\alpha^- \otimes X_\alpha^- \otimes 1_u\}\hat{\otimes} v_0 \quad \forall \hat{\Lambda},
\end{aligned} \tag{3.10b}
$$

$$
\begin{aligned}
{}_3v_s^{3\alpha} = \{&\hat{\Lambda}^2(X_\alpha^-)^3 \otimes 1_u \otimes 1_u \\
&- 3\hat{\Lambda}(\hat{\Lambda} - 2)(X_\alpha^-)^2 \otimes X_\alpha^- \otimes 1_u \\
&+ 2(\hat{\Lambda} - 1)(\hat{\Lambda} - 2) X_\alpha^- \otimes X_\alpha^- \otimes X_\alpha^-\}\hat{\otimes} v_0 \quad \forall \hat{\Lambda},
\end{aligned} \tag{3.10c}
$$

$$
\begin{aligned}
{}_3v_s^{4\alpha} = \{&\hat{\Lambda}(\hat{\Lambda} - 1)(X_\alpha^-)^4 \otimes 1_u \otimes 1_u \\
&- 4(\hat{\Lambda} - 1)(\hat{\Lambda} - 3)(X_\alpha^-)^3 \otimes X_\alpha^- \otimes 1_u \\
&+ 3(\hat{\Lambda} - 2)(\hat{\Lambda} - 3)(X_\alpha^-)^2 \otimes (X_\alpha^-)^2 \otimes 1_u\}\hat{\otimes} v_0 \quad \forall \hat{\Lambda},
\end{aligned} \tag{3.10d}
$$

$$
\begin{aligned}
{}_3v_s'^{4\alpha} = \{&(X_\alpha^-)^4 \otimes 1_u \otimes 1_u + 8(X_\alpha^-)^3 \otimes X_\alpha^- \otimes 1_u \\
&+ 12(X_\alpha^-)^2 \otimes X_\alpha^- \otimes X_\alpha^-\}\hat{\otimes} v_0, \quad \hat{\Lambda} = 1,
\end{aligned} \tag{3.10e}
$$

$$
\begin{aligned}
{}_3v_s^{5\alpha} = \{&\hat{\Lambda}^2(\hat{\Lambda} - 1)(X_\alpha^-)^5 \otimes 1_u \otimes 1_u \\
&- 5\hat{\Lambda}(\hat{\Lambda} - 1)(\hat{\Lambda} - 4)(X_\alpha^-)^4 \otimes X_\alpha^- \otimes 1_u \\
&+ 2\hat{\Lambda}(\hat{\Lambda} - 3)(\hat{\Lambda} - 4)(X_\alpha^-)^3 \otimes (X_\alpha^-)^2 \otimes 1_u \\
&+ 8(\hat{\Lambda} - 1)(\hat{\Lambda} - 3)(\hat{\Lambda} - 4)(X_\alpha^-)^3 \otimes X_\alpha^- \otimes X_\alpha^- \\
&- 6(\hat{\Lambda} - 2)(\hat{\Lambda} - 3)(\hat{\Lambda} - 4)(X_\alpha^-)^2 \otimes (X_\alpha^-)^2 \otimes X_\alpha^-\}\hat{\otimes} v_0 \quad \forall \hat{\Lambda},
\end{aligned} \tag{3.10f}
$$

$$
{}_3v_s'^{5\alpha} = \{2(X_\alpha^-)^3 \otimes (X_\alpha^-)^2 u \otimes 1_u
$$

$$- (X_\alpha^-)^3 \otimes X_\alpha^- \otimes X_\alpha^- \}\hat{\otimes} v_0, \quad \hat{\Lambda} = 2, \tag{3.10g}$$

$$\begin{aligned}
{}_3v_{\rm s}^{6\alpha} = \{ & \hat{\Lambda}(\hat{\Lambda}-1)(\hat{\Lambda}-2)(X_\alpha^-)^4 \otimes (X_\alpha^-)^2 \otimes 1_{\rm u} \\
& - (\hat{\Lambda}-1)^2(\hat{\Lambda}-2)(X_\alpha^-)^4 \otimes X_\alpha^- \otimes X_\alpha^- \\
& - \hat{\Lambda}(\hat{\Lambda}-1)(\hat{\Lambda}-3)(X_\alpha^-)^3 \otimes (X_\alpha^-)^3 \otimes 1_{\rm u} \\
& + 2(\hat{\Lambda}-1)(\hat{\Lambda}-2)(\hat{\Lambda}-3)(X_\alpha^-)^3 \otimes (X_\alpha^-)^2 \otimes X_\alpha^- \\
& - (\hat{\Lambda}-2)^2(\hat{\Lambda}-3)(X_\alpha^-)^2 \otimes (X_\alpha^-)^2 \otimes (X_\alpha^-)^2 \}\hat{\otimes} v_0 \quad \forall \hat{\Lambda},
\end{aligned} \tag{3.10h}$$

$$\begin{aligned}
{}_3v_{\rm s}^{\prime 6\alpha} = \{ & \hat{\Lambda}(\hat{\Lambda}-1)(\hat{\Lambda}-2)(X_\alpha^-)^6 \otimes 1_{\rm u} \otimes 1_{\rm u} \\
& - 6(\hat{\Lambda}-1)(\hat{\Lambda}-2)(\hat{\Lambda}-5)(X_\alpha^-)5 \otimes X_\alpha^- \otimes 1_{\rm u} \\
& + 15(\hat{\Lambda}-2)(\hat{\Lambda}-4)(\hat{\Lambda}-5)(X_\alpha^-) \otimes (X_\alpha^-)^2 \otimes 1_{\rm u} \\
& - 10(\hat{\Lambda}-3)(\hat{\Lambda}-4)(\hat{\Lambda}-5)(X_\alpha^-)^3 \otimes (X_\alpha^-)^3 \otimes 1_{\rm u}\}\hat{\otimes} v_0 \quad \forall \hat{\Lambda}.
\end{aligned} \tag{3.10i}$$

We give those examples in order to point out some new features appearing for tri-Verma modules in comparison with the bi-Verma modules:

- independently of $\Lambda(H)$ there exists a singular vector at any level $n\alpha$, except the lowest $n = 1$, while for bi-Verma modules singular vectors at odd levels exist only for special values of $\Lambda(H)$;
- there exists more than one singular vector at any fixed level $n\alpha$ for $n \geq 6$ and arbitrary $\Lambda(H)$. For special values of $\Lambda(H)$ there exists a second singular vector for $n = 4, 5$.

Similar facts hold for k-Verma modules for $k > 3$. These questions will be considered in another publication. In the present paper we would like to demonstrate in the following sections examples sharing the usefulness of these modules.

4. Multilinear intertwining differential operators

4.1

We start here by sketching the procedure of [10] for construction of *linear* intertwining differential operators which we generalize in Section 4.2 for *multilinear* intertwining differential operators. Let G be a semisimple Lie group and let $\mathcal{G}$ denote its Lie algebra. (Note that the procedure works in the same way for a reductive Lie group, since only its semisimple subgroup is essential for the construction of the intertwining differential operators. We restrict to semisimple groups for simplicity. For more technical simplicity one may assume that in addition G is linear and connected.) Let $G = K\,A_0\,N_0$ be an Iwasawa decomposition of G, where K is the maximal compact subgroup of G, A_0 is abelian simply connected, the so-called vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. (M_0 has the structure $M_0 = M_0^d\, M_0^r$, where M_0^d is a finite group, M_0^r is reductive with the same Lie algebra as M_0.) Then $P_0 = M_0\,A_0\,N_0$ is called a *minimal parabolic subgroup* of G. A *parabolic subgroup* of G is any subgroup which is isomorphic to a subgroup $P = MAN$ such that: $P_0 \subset P \subset G$, $M_0 \subset M$, $A_0 \supset A$, $N_0 \supset N$. (Note that in the above considerations every subgroup N may be exchanged with its Cartan conjugate $\tilde{N}$.) The number of non-conjugate parabolic subgroups (counting also the case $P = G = M$) is 2^{r_0}, $r_0 = \dim A_0$.

Parabolic subgroups are important because the representations induced from them generate all admissible irreducible representations of G [23,25]. In fact, for this it is enough to use only the so-called *cuspidal* parabolic subgroups, singled out by the condition that rank M = rank $M \cap K$; thus M has discrete series representations.

Let P be a cuspidal parabolic subgroup and let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ or the so-called limit of a discrete series representation (cf. [21]). Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, where $\mathcal{A}$ is the Lie algebra of A. We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G. (These are called *generalized principal series representations* (or limits thereof) in [21].) Consider now the space of functions

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = \mathrm{e}^{\nu(H)} D^\mu(m^{-1})\mathcal{F}(g)\}, \tag{4.1}$$

where $g \in G$, $m \in M$, $a = \exp(H)$, $H \in \mathcal{A}$, $n \in N$. The special property of the functions of $\mathcal{C}_\chi$ is called *right covariance* [10] (or *equivariance*). It is well known that $\mathcal{C}_\chi$ can be thought of as the space of smooth sections of the homogenous vector bundle (called also vector G-bundle) with base space G/P and fibre V_μ (which is an associated bundle to the principal P-bundle with total space G).

Then the elementary representation (ER) $\mathcal{T}^\chi$ acts in $\mathcal{C}_\chi$, as the left regular representation (LRR), by

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \tag{4.2}$$

(In practice, the same induction is used with non-discrete series representations of M and also with non-cuspidal parabolic subgroups.) One can introduce in $\mathcal{C}_\chi$ a Fréchet space topology or complete it to a Hilbert space (cf. [21]). The ERs differ from the LRR (which is highly reducible) by the specific representation spaces $\mathcal{C}_\chi$. In contrast, the ERs are generically irreducible. The reducible ERs from a measure zero set in the space of the representation parameters μ, ν. (Reducibility here is topological in the sence that there exists a non-trivial (closed) invariant subspace.) Next we note that in order to obtain the intertwining differential operators one may consider the infinitesimal version of (4.2), namely

$$(\mathcal{L}^\chi(X)\mathcal{F})(g) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\exp(-tX)g)|_{t=0}, \quad X \in \mathcal{G}. \tag{4.3}$$

The feature of the ERs which makes them important for our considerations in [10] and here is a highest weight module structure associated with them. (It would be lowest weight module structure, if one exchanges N with $\tilde{N}$, as is actually done in [10].) Let $\mathcal{G}_{\mathrm{c}} = \mathcal{G}$ if $\mathcal{G}$ is complex or real split, otherwise let $\mathcal{G}_{\mathrm{c}}$ be the complexification of $\mathcal{G}$ with triangular decomposition: $\mathcal{G}_{\mathrm{c}} = \mathcal{G}_{\mathrm{c}}^+ \oplus \mathcal{H}_{\mathrm{c}} \oplus \mathcal{G}_{\mathrm{c}}^-$. Further introduce the right action of $\mathcal{G}_{\mathrm{c}}$ by the standard formula:

$$(\hat{X}\mathcal{F})(g) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(g\exp(tX))|_{t=0}, \tag{4.4}$$

where, $X \in \mathcal{G}_c, \mathcal{F} \in \mathcal{C}_\chi, g \in G$, which is defined first for $X \in \mathcal{G}$ and then is extended to $\mathcal{G}_c$ by linearity. Note that this action takes $\mathcal{F}$ out of $\mathcal{C}_\chi$ for some X but that is exactly why it is used for the construction of the intertwining differential operators.

We illustrate the highest weight module structure in the case of the minimal parabolic subgroup. In that case $M = M_0$ is compact and V_μ is finite dimensional. Consider first the case when M_0 is non-abelian. Let v_0 be the highest weight vector of V_μ. Now we can introduce $\mathbb{C}$-valued realization $\tilde{\mathcal{C}}_\chi$ of the space $\mathcal{C}_\chi$ by the formula:

$$\varphi(g) \equiv \langle v_0, \mathcal{F}(g) \rangle, \tag{4.5}$$

where $\langle\ ,\ \rangle$ is the M_0-invariant scalar product in V_μ. On these functions the counterpart $\tilde{\mathcal{T}}^\chi$ of the LRR (4.2), its infinitesimal form (4.3) and the right action of $\mathcal{G}_c$ are defined as inherited from the functions $\mathcal{F}$:

$$\tilde{\mathcal{T}}^\chi(g)\varphi \doteq \langle v_0, \mathcal{T}^\chi(g)\mathcal{F} \rangle, \tag{4.6a}$$

$$\tilde{\mathcal{L}}^\chi(X)\varphi \doteq \langle v_0, \mathcal{L}^\chi(X)\mathcal{F} \rangle, \tag{4.6b}$$

$$\tilde{X}\varphi \doteq \langle v_0, \hat{X}\mathcal{F} \rangle. \tag{4.6c}$$

In the geometric language we have replaced the homogeneous vector bundle with base space G/P and fibre V_μ with a line bundle again with base space G/P (also associated to the principal P-bundle with total space G). The functions φ can be thought of as smooth sections of this line bundle. If M_0 is abelian or discrete then V_μ is one-dimensional and $\tilde{\mathcal{C}}_\chi$ coincides with $\mathcal{C}_\chi$; so we set $\varphi = \mathcal{F}$. Part of the main result of [10] is:

Proposition 3. *The functions of the* $\mathbb{C}$*-valued realization* $\tilde{\mathcal{C}}_\chi$ *of the ER* $\mathcal{C}_\chi$ *satisfy*

$$\hat{X}\varphi = \Lambda(X)\varphi, \quad X \in \mathcal{H}_c, \tag{4.7a}$$

$$\hat{X}\varphi = 0, \quad X \in \mathcal{G}_c^+, \tag{4.7b}$$

where $\Lambda = \Lambda_\chi \in (\mathcal{H}_c)^*$ *is built canonically from* χ *and contains all the information from* χ*, except about the character* ϵ *of the finite group* M_0^d*).*

Now we note that conditions (4.7) are the defining conditions for the highest weight vector of a highest weight module (HWM) over $\mathcal{G}_c$ with highest weight Λ, in particular, of a Verma module with this highest weight, cf. (2.1).

Let the signature χ of an ER be such that the corresponding $\Lambda = \Lambda_\chi$ satisfies (2.8) for some $\beta \in \Delta^+$ and some $m \in \mathbb{N}$. If β is a real root, then some conditions are imposed on the character ϵ representing the finite group M_0^d [28]. Then there exists an intertwining differential operator [10]:

$$\mathcal{D}^{m\beta} : \tilde{\mathcal{C}}_\chi \longrightarrow \tilde{\mathcal{C}}_{\chi'}, \tag{4.8a}$$

$$\mathcal{D}^{m\beta} \circ \tilde{\mathcal{T}}^\chi(g) = \tilde{\mathcal{T}}^{\chi'}(g) \circ \mathcal{D}^{m\beta} \quad \forall_g \in G, \tag{4.8b}$$

where χ' is uniquely determined so that $\Lambda' = \Lambda_{\chi'} = \Lambda - m\beta$.

The important fact is that (4.8) is explicitly given by [10]:

$$\mathcal{D}^{m\beta}\varphi(g) = P^{m\beta}(\hat{X}_1^-, \ldots, \hat{X}_l^-)\varphi(g), \tag{4.9}$$

where $P^{m\beta}$ is the same polynomial as in (2.4) and $\hat{X}_j^-$ denotes the action (4.6b). One important technical simplification is that the intertwining differential operators (4.9) are *scalar* operators since they intertwine two *line* bundles $\tilde{C}_\chi, \tilde{C}_{\chi'}$.

4.2

Now we generalize the above sketched construction of [10] to *multilinear intertwining differential operators*. We have the following.

Proposition 4. *Let the signature χ of an ER be such that the k-Verma module ${}_kV^\Lambda$ with highest weight $\Lambda = \Lambda_\chi$ has a singular vector ${}_kv_s^\mu \in {}_kV_\mu^\Lambda$, i.e.,* (3.1) *is satisfied for some $\mu \in \Gamma^+$. Let us denote*

$${}_kv_s^\mu = {}_k\mathcal{P}^\mu \hat{\otimes} v_0, \tag{4.10}$$

where ${}_k\mathcal{P}^\mu \in \mathcal{S}_k(\mathcal{G}^-)$ is some concrete polynomial as in (2.18c). *Then there exists a multilinear intertwining differential operator which we denote by ${}_k\mathcal{I}_\mu^\Lambda$ such that*

$${}_k\mathcal{I}_\mu^\Lambda : \underbrace{\varphi \otimes \cdots \otimes \varphi}_{k} \to \psi, \quad \varphi \in \tilde{C}_\chi, \quad \psi \in \tilde{C}_{\chi'}, \tag{4.11}$$

$$\begin{aligned} &{}_k\mathcal{I}_\mu^\Lambda \circ \sum_{j=1}^{k} \underbrace{1_u \otimes \cdots \otimes \tilde{\mathcal{L}}^\chi(X) \otimes \cdots \otimes 1_u}_{k} \\ &\quad = \tilde{\mathcal{L}}^{\chi'}(X) \circ {}_k\mathcal{I}_\mu^\Lambda \quad \forall X \in \mathcal{G}, \end{aligned} \tag{4.12}$$

where χ' is uniquely determined (*up to the representation parameters of the discrete subgroup M^d*) *so that $\Lambda' = \Lambda\chi' = k\Lambda - \mu$. The operator is given explicitly by the same polynomial as in* (4.10), *i.e.,*

$${}_k\mathcal{I}_\mu^\Lambda \underbrace{(\varphi \otimes \cdots \otimes \varphi)}_{k} = {}_k\widehat{\mathcal{P}}^\mu \underbrace{(\varphi \otimes \cdots \otimes \varphi)}_{k}, \tag{4.13}$$

where the hat on ${}_k\mathcal{P}^\mu$ symbolizes the right action (4.6b), *the explicit action of a typical term of ${}_k\mathcal{P}^\mu$ being* (*cf.* (2.18c)):

$$\begin{aligned} &\{X_{\beta_1^1}^- \cdots X_{\beta_{n_1}^1}^- \otimes \widehat{\cdots} \otimes X_{\beta_1^k}^- \cdots X_{\beta_{n_k}^k}^-\} \underbrace{\varphi \otimes \cdots \otimes \varphi}_{k} \\ &\quad = (\hat{x}_{\beta_1^1} \cdots \hat{X}_{\beta_{n_1}^1}^- \varphi) \cdots (\hat{X}_{\beta_1^k}^- \cdots \hat{X}_{\beta_1^k}^- \cdots \hat{X}_{\beta_{n_k}^k}^- \varphi). \end{aligned} \tag{4.14}$$

Proof. Completely anagolous to the case $k = 1$ (cf. [10]). □

Remark 2. The analogue of the intertwining property (4.12) on the group level, i.e.,

$$ {}_k\mathcal{I}^{\Lambda}_{\mu} \circ \underbrace{\tilde{T}^{\chi}(g) \otimes \cdots \otimes \tilde{T}^{\chi}(g)}_{k} = \tilde{T}^{\chi'}(g) \circ {}_k\mathcal{I}^{\Lambda}_{\mu} \quad \forall g \in G, \tag{4.15} $$

will hold, in general, for less values of Λ than (4.12). This is in sharp contrast with the $k = 1$ case, where there is no difference in this respect. An additional feature on the group level common for all $k \geq 1$ is that some discrete representation parameters of χ, not represented in Λ, get fixed.

Remark 3. Let us stress that since we have realized arbitrary representations in the spaces of scalar-valued functions φ then also the intertwining differential operators are *scalar* operators in all cases – geometrically speaking, these operators intertwine (tensor products of) line bundles. This simplicity may be contrasted with the proliferation of tensor indices in the approaches relying on Weyl's $SO(n)$ polynomial invariant theory [20], cf., e.g., [3,13], where $G = SO(n+1, 1)$, $M = M_0 = SO(n)$, $\dim A = 1$.

Finally we should mention that the simplest formulae are obtained of one restricts the functions to the conjugate to N subgroup $\tilde{N}$ [10]:

$$ C_{\chi} \doteq \{\phi = R\varphi \mid \varphi \in \tilde{C}_{\chi}, (R\varphi)(\tilde{n}) \doteq \varphi(\tilde{n}), \tilde{n} \in \tilde{N}\}. \tag{4.16} $$

Clearly, the elements of C_{χ} and consequently $\tilde{C}_{\chi}$ are almost determined by their values on $\tilde{N}$ because of right covariance (4.1) and because, up to a finite number of submanifolds of strictly lower dimension, every element of G belongs to $\tilde{N}MAN$. The latter are open dense submanifolds of G of the same dimension forming non-global Bruhat decompositions of G. Connectedly, $\tilde{N}$ is an open dense submanifold of G/P.

The ER T^{χ} acts in this space by

$$ \begin{aligned} &(T^{\chi}(g)\phi)(\tilde{n}) = e^{\nu(H)} D^{\mu}(m)^{-1}\phi(\tilde{n}'), \\ &D^{\mu}(m)\phi(\tilde{n}) = D^{\mu}(m)\varphi(\tilde{n}) = \langle v_0, D^{\mu}(m)\mathcal{F}(\tilde{n})\rangle, \end{aligned} \tag{4.17} $$

where $g \in G$, $\tilde{n}, \tilde{n}' \in \tilde{N}$, $m \in M$, $a = \exp(H)$, $H \in \mathcal{A}$, and we have used the Bruhat decomposition $g^{-1}\tilde{n} = \tilde{n}'man (n \in N)$. (The transformation can also be defined seprately for $g^{-1}\tilde{n} \notin \tilde{N}MAN$ and there exists smooth passage from (4.17) to these expressions. This is related to the passage between different coordinate patches of G/P.) One may easily check that the restriction operator R intertwines the two representations, i.e.,

$$ T^{\chi}(g)R = R\tilde{T}^{\chi}(g) \quad \forall g \in G. \tag{4.18} $$

5. Bilinear operators for $SL(n, R)$ and $SL(n, C)$

5.1

In this section we restrict ourselves to the case $G = SL(n, \mathbb{R})$, mentioning also which results are extendable to $SL(n, \mathbb{C})$. We use the following matrix representations for G, its

Lie algebra $\mathcal{G}$ and some subgroups and subalgebras:

$$G = SL(n,\mathbb{R}) = \{g \in gl(n,\mathbb{R}) \mid \det g = 1\}, \tag{5.1a}$$
$$\mathcal{G} = sl(n,\mathbb{R}) = \{X \in gl(n,\mathbb{R}) \mid \mathrm{tr}\, X = 0\}, \tag{5.1b}$$
$$K = SO(n) = \{g \in SL(n,\mathbb{R}) \mid gg^{\mathrm{t}} = g^{\mathrm{t}}g = 1_n\}, \tag{5.1c}$$
$$\mathcal{A}_0 = \{X \in sl(n,\mathbb{R}) \mid X \text{ diagonal}\}, \tag{5.1d}$$
$$A_0 = \exp(\mathcal{A}_0) = \{g \in SL(n,\mathbb{R}) \mid g \text{ diagonal}\}, \tag{5.1e}$$
$$\begin{aligned} M_0 &= \{m \in K \mid ma = ma \forall a \in A_0\} \\ &= \{m = \mathrm{diag}(\delta_1, \delta_2 \ldots, \delta_n) \mid \delta_k = \pm, \\ &\qquad \delta_1\delta_2\cdots\delta_n = 1\} = M_0^d, \end{aligned} \tag{5.1f}$$
$$\mathcal{N}_0 = \{X = (a_{ij}) \in gl(n,\mathbb{R}) \mid a_{ij} = 0, i \geq j\}, \tag{5.1g}$$
$$\begin{aligned} N_0 &= \exp(\mathcal{N}_0) \\ &= \{X = (a_{ij}) \in gl(n,\mathbb{R}) \mid a_{ii} = 1, a_{ij} = 0, i > j\}, \end{aligned} \tag{5.1h}$$
$$\tilde{\mathcal{N}}_0 = \{X = (a_{ij}) \in gl(n,\mathbb{R}) \mid a_{ij} = 0, i \leq j\}, \tag{5.1i}$$
$$\begin{aligned} \tilde{N}_0 &= \exp(\tilde{\mathcal{N}}_0) \\ &= \{X = (a_{ij}) \in gl(n,\mathbb{R}) \mid a_{ii} = 1, a_{ij} = 0, i < j\}. \end{aligned} \tag{5.1j}$$

Since the algebra $sl(n,\mathbb{R})$ is maximally split then the Bruhat decomposition with the minimal parabolic:

$$\mathcal{G} = sl(n,\mathbb{R}) = \tilde{N}_0 \oplus \mathcal{A}_0 \oplus \tilde{\mathcal{N}}_0 \tag{5.2}$$

may be viewed as a restriction from $\mathbb{C}$ to $\mathbb{R}$ of the triangular decomposition of its complexification

$$\mathcal{G}^{\mathbb{C}} = sl(n,\mathbb{C}) = \mathcal{G}_+^{\mathbb{C}} \oplus \mathcal{H} \oplus \mathcal{G}_-^{\mathbb{C}}. \tag{5.3}$$

Accordingly, we may use for both cases the same Chevalley basis consisting of the $3(n-1)$ generators X_i^+, X_i^-, H_i given explicitly by

$$\begin{aligned} &X_i^+ = E_{i,i+1}, \qquad X_i^- = E_{i+1,i}, \qquad H_i = E_{ii} - E_{i=1,i+1}, \\ &\quad i = 1, \ldots, n-1, \end{aligned} \tag{5.4}$$

where E_{ij} are the standard matrices with 1 on the intersection of the ith row and jth column and zeros everywhere else. Note that X_i^+, X_i^-, H_i, respectively, generate $\tilde{N}_0$, $\tilde{\mathcal{N}}_0$, $\mathcal{A}_0$, respectively, over $\mathbb{R}$ and $\mathcal{T}_+^{\mathbb{C}}$,$\mathcal{G}_-^{\mathbb{C}}$, $\mathcal{H}$, respectively over $\mathbb{C}$.

5.2

We consider induction from the minimal parabolic case, i.e., $P = M_0A_0N_0$. The characters of the discrete group $M_0 = M_0^d$ are labelled by the signature: $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$, $\epsilon_k = 0, 1$:

$$\mathrm{ch}_\epsilon(m) = \mathrm{ch}_\epsilon(\delta_1, \delta_2, \ldots, \delta_n) \doteq \prod_{k=1}^{n-1} (\delta_k)^{\epsilon_k}, \quad m \in M_0. \tag{5.5}$$

The (non-unitary) characters $v \in \mathcal{A}_0^*$ of A_0 are labelled by $c_k \in \mathbb{C}$, $k = 1, \ldots, n-1$, which is the value of v on $H_k = \mathrm{diag}\,(0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathcal{A}_0$ (with the unity on kth place), $k = 1, \ldots, n-1$, i.e., $c_k = v(H_k)$:

$$\begin{aligned} \mathrm{ch}_c(a) &= \mathrm{ch}_c\left(\exp \sum_k t_k H_k\right) \doteq \exp \sum_k t_k v(H_k) \\ &= \exp \sum_k t_k c_k = \prod_k \hat{a}_k^{c_k}, \\ a &= \prod_k a_k \in A_0, \\ a_k &= \exp t_k H_k \in A_0, \quad t_k, \hat{a}_k = \exp t_k \in \mathbb{R}. \end{aligned} \tag{5.6}$$

Thus, the right covariance property (4.1) is

$$\mathcal{F}(gman) = \prod_{k=1}^{n-1} (\delta_k)^{\epsilon_k} \hat{a}_k^{c_k} \mathcal{F}(g), \tag{5.7}$$

and in this case we have scalar function, i.e., $\varphi = \mathcal{F}$ because M_0 is discrete. Thus, the ER acts on the restricted functions as (cf. (4.17)):

$$(T^{c,\epsilon}(g)\phi)(\tilde{n}) = \prod_{k=1}^{n-1} (\delta_k)^{\epsilon_k} \hat{a}_k^{c_k} \phi(\tilde{n}') \tag{5.8}$$

(note that $\delta_k = (\delta_k)^{-1}$). The functions ϕ depend on the $\frac{1}{2}n(n-1)$ non-trivial elements of the matrices of $\tilde{\mathcal{N}}_0$. For further use those will be denoted by z_j^i, i.e., for $\tilde{n} \in \tilde{\mathcal{N}}_0$ we have $\tilde{n} = (a_{ij})$ with $a_{ij} = z_j^i$ for $i > j$, cf. (5.1j).

The correspondence between the ER with signature $\chi = [c, \epsilon]$ and the highest weight Λ, used in the general construction of Section 4, here is very simple [10]: $\Lambda = -v$, so that $\Lambda(H) = -v(H)$. Further, we recall that the root system of $sl(n, \mathbb{C})$ is given by roots: $\pm\alpha_{ij}$, $i < j$, so that $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $i+1 < j$ and $\alpha_{i,i+1} = \alpha_i$, where α_i, $i = 1, \ldots, n-1$, are the simple roots with non-zero scalar products: $(\alpha_i, \alpha_i) = \alpha_i(H_i) = 2$, $(\alpha_i, \alpha_{i+1}) = \alpha_i(\alpha_{i+1}) = -1$, then $a_i^\vee = \alpha_i$. Then we use $(v, \alpha_i) = v(H_i) = c_i$.

We need also the infinitesimal version of (5.8):

$$\tilde{\mathcal{L}}^c(Y)\phi(\tilde{n}) \doteq \left(\frac{\mathrm{d}}{\mathrm{d}t}(T^{c,\epsilon}(\exp tY)\phi)(\tilde{n})\right)_{|t=0}, \quad Y \in \mathcal{G}, \tag{5.9}$$

which we give for the Chevalley generators (5.4) explicitly:

$$\tilde{\mathcal{L}}^c(X_i^+) = z_i^{i+1}\left(c_i + \sum_{k=i=1}^{n} N_i^k - \sum_{k=i+2}^{n} N_{i+1}^k\right)$$

$$- \sum_{s=1}^{i-1} z_s^{i+1} D_s^i + \sum_{k=i+2}^{n} z_i^k D_{i+1}^k, \tag{5.10a}$$

$$\tilde{\mathcal{L}}^c(X_i^-) = -D_i^{i+1} - \sum_{s=1}^{i-1} z_s^i D_s^{i+1}, \tag{5.10b}$$

$$\tilde{\mathcal{L}}^c(H_i) = c_i + \sum_{k=i+1}^{n} N_i^k - \sum_{k=i+2}^{n} N_{i+1}^k$$
$$- \sum_{s=1}^{i-1} N_s^i + \sum_{s=1}^{i} N_s^{i+1}, \tag{5.10c}$$

where $D_j^i \equiv \partial/\partial z_j^i$, $N_j^i \equiv z_j^i(\partial/\partial z_j^i)$, and we are using the convention that when the lower summation limit is bigger than the higher summation limit then the sum is zero.

We also need the right action (4.4) for the lowering generators which on the restricted functions is given explicitly by

$$\hat{X}_i^- \phi(\tilde{n}) = \left(D_i^{i+1} + \sum_{k=i+2}^{n} z_{i+1}^k D_i^k \right) \phi(\tilde{n}). \tag{5.11}$$

Naturally the signature ϵ representing the discrete subgroup $M = M_d^0$ is not present in (5.10) and (5.11). Thus, formulae (5.10) and (5.11) are valid also for the holomorphic ERs of $SL(n, \mathbb{C})$.

Now to obtain explicit examples of multilinear intertwining differential operators it remains to substitute formula (5.11) in the corresponding formulae for the singular vectors of the k-Verma modules. We note that often a singular vector will produce many intertwining differential operators. For example each formula valid for any simple root will produce $n-1$ formula, each formula valid for roots as $\alpha_1 + \alpha_2$ will produce $n-2$ formulae for each $\alpha_i + \alpha_{i+1}$. To save space we shall not write these formulae except in a few examples in the cases $n = 2$ and $n = 3$.

5.3

Now we restrict ourselves to the case $G = SL(2, \mathbb{R})$. We denote: $x = z_1^2$, $c = c_1$, $\epsilon = \epsilon_1$, $X^\pm = X_1^\pm$, $H = H_1$. We start with bilinear intertwining differential operators, $k = 2$. We combine Propositions 2 and 4. For $\mathcal{G} = sl(2, \mathbb{R})$ Proposition 2 gives *all singular vectors* of bi-Verma modules since all weights in Γ^+ are of the form $\mu = n\alpha$, $n \in \mathbb{N}$. Thus we have:

Theorem 1. *All bilinear intertwining differential operators for the case of* $\mathcal{G} = sl(2, \mathbb{R})$ *are given by the formula:*

$${}_2\mathcal{I}_{n\alpha}^{\Lambda}(\phi) = \sum_{j=0}^{[n/2]} \gamma_{nj}^{\Lambda} \phi^{(n-j)} \phi^{(j)}, \tag{5.12}$$

where $\phi^{(p)} \doteq (\partial_x)^p \phi(x)$, $\partial_x \doteq \partial/\partial x$, *and the coefficients* γ^{Λ}_{nj} *are given in Proposition 2. The intertwining property is*

$$\begin{aligned} &{}_2\mathcal{I}^{\Lambda}_{n\alpha} \circ (\tilde{\mathcal{L}}^c(X) \otimes 1_{\mathrm{u}} + 1_{\mathrm{u}} \otimes \tilde{\mathcal{L}}^c(X)) \\ &\quad = \tilde{\mathcal{L}}^{c}(X) \circ_2 \mathcal{I}^{\Lambda}_{n\alpha} \quad \forall X \in \mathcal{G}, \\ &\quad c = -\Lambda(H), \quad c' = 2(c+n). \end{aligned} \tag{5.13}$$

Proof. Follows from the elementary combination of Propositions 2 and 4 in the present setting. In particular, $\Lambda' = 2\Lambda - n\alpha$, $c' = -\Lambda'(H) = -2\Lambda(H) + n\alpha(H) = 2(c+n)$. □

As we mentioned, the corresponding intertwining property on the group level is restricting the values of Λ and, as for $k = 1$, of some discrete parameters not represented in Λ. In the present case we have:

Theorem 2. *All bilinear intertwining differential operators for the case of* $G = SL(2,\mathbb{R})$ *are given by formulae* (5.12) *and* (3.3) *with 'integer' highest weight:* $\Lambda(H) = p \in \mathbb{Z}$. *The intertwining property is*

$$\begin{aligned} &{}_2\mathcal{I}^{\Lambda}_{n\alpha} \circ (T^{c,\epsilon}(g) \otimes T^{c,\epsilon}(g)) = T^{c',\epsilon'}(g) \circ_2 \mathcal{I}^{\Lambda}_{n\alpha} \quad \forall g \in G, \\ &\quad c = -\Lambda(H) = -p, \qquad c' = 2(c+n) = 2(n-p), \\ &\quad \epsilon = \epsilon' = p (mod\, 2). \end{aligned} \tag{5.14}$$

Proof. Follows by using Theorem 1 and checking (5.14) for

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which sends $x \neq 0$ into $-1/x$. □

Thus, the signatures of the two intertwined spaces coincide and are determined by the parameter p.

Next we consider the example of *invariant* functions ϕ, i.e., functions for which the transformation law (5.8) has no multipliers – this happens iff $c = \epsilon = 0$. For these functions the bilinear intertwining differential operators are given by the special case of Theorem 2 when $\Lambda(H) = p = 0$, which by (3.3) further restricts n to be even or $n = 1$. Formula (5.12) with (3.3) substituted simplifies to

$$\begin{aligned} {}_z\mathcal{I}^{0}_{n\alpha}(\phi) = &\sum_{j=1}^{n/2} (-1)^{j-1}(1 - \tfrac{1}{2}\delta_j, n/2) \\ &\times \frac{n-j}{n(n-1)} \binom{n}{j} \binom{n-1}{j-1} \phi^{(n-j)} \phi^{(j)}, \quad n \in 2\mathbb{R}, \end{aligned} \tag{5.15}$$

$${}_2\mathcal{I}^{0}_{\alpha}(\phi) = \phi\partial_x\partial = \phi\phi' \tag{5.16}$$

and in addition we have fixed the constatn γ_0 for later convenience. Let us write out the first several cases of (5.15):

$$ {}_2\mathcal{I}^0_{2\alpha}(\phi) = \tfrac{1}{2}(\phi')^2, \tag{5.17a} $$

$$ {}_2\mathcal{I}^0_{4\alpha}(\phi) = \phi'''\phi' - \tfrac{3}{4}(\phi'')^2, \tag{5.17b} $$

$$ {}_2\mathcal{I}^0_{6\alpha}(\phi) = \phi^{(5)}\phi' - 10\phi^{(4)}\phi'' + 10(\phi''')^2, \tag{5.17c} $$

where (standardly) $\phi' \equiv \partial_x\phi = \phi^{(1)}$, $\phi'' \equiv \partial_x^2\phi = \phi^{(2)}$, $\phi''' \equiv \partial_x^3\phi = \phi^{(3)}$. Note that (5.17b) (i.e., (5.15) for $n = 4$) was already given (1.7). We give now two important technical statements.

Lemma 1. *For $n > 2$ the (formal) substitution $\phi(x) \to (\alpha x - \gamma)/(\delta - \beta_x)$ in the intertwining differential operators* (5.15) *gives zero:*

$$ {}_2\mathcal{I}^0_{n\alpha}(\phi_0) = 0, \quad \phi_0(x) \equiv \frac{\alpha x - \gamma}{\delta - \beta x}, \quad n \in 2 + 2\mathbb{N}. \tag{5.18} $$

Proof. Follows from the direct substitution. In the calculations one uses the fact

$$ \partial_x^m \frac{\alpha x - \gamma}{\delta - \beta x} = \frac{(-1)^m m! \beta^{m-1}}{(\delta - \beta x)^{m+1}}, \quad m \in \mathbb{N}. \tag{5.19} $$

After the substitution of (5.19) in (5.15) the resulting expression is proportional to $(1-1)^{n-2}$ which is zero for $n > 2$ (the latter making clear why the lemma is not valid for $n = 2$). □

Lemma 2. *Let $\phi, \psi \in \mathrm{Diff}_0 S^1$, the group of orientation preserving diffeomorphisms of the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then we have*

$$ {}_2\mathcal{I}^0_{n\alpha}(\phi \circ \psi) = (\psi')^n_2\mathcal{I}^0_{n\alpha}(\phi), \quad n = 1, 2, \tag{5.20} $$

$$ \begin{aligned} {}_2\mathcal{I}^0_{n\alpha}(\phi \circ \psi) = {} & (\psi')^n_2\mathcal{I}^0_{n\alpha}(\phi) + (\phi')^2_2\mathcal{I}^0_{n\alpha}(\phi) \\ & + P_n(\phi, \psi), \quad n \in 2 + c\mathbb{N} \end{aligned} \tag{5.21a} $$

$$ P_4(\phi, \psi) = 0, \tag{5.21b} $$

$$ P_n(\phi, \phi_0) = 0, \quad n \in 4 + 2\mathbb{N}. \tag{5.21c} $$

Proof. For (5.20) and (5.21) (5.21b) this is just substitution. Further we note that

$$ {}_2\mathcal{I}^0_{n\alpha}(\phi \circ \phi_0) = (\phi_0')^n_2\mathcal{I}^0_{n\alpha}(\phi), \quad \phi_0(x) = \frac{\alpha x - \gamma}{\delta - \beta x} \tag{5.22} $$

is just the intertwining property of ${}_2\mathcal{I}^0_{n\alpha}$ and then (5.21c) follows because of Lemma 1. □

Remark 4. We give an example from Lemma 2

$$ \begin{aligned} P_6(\phi, \psi) = {} & 10(\psi')^2(3\phi'''\phi' - 4(\phi'')^2)_2\mathcal{I}^0_{4\alpha}(\psi) \\ & - 5\phi''\phi'(\psi^{(4)}(\psi')^2 - 6\psi'''\psi''\psi' + 6(\psi'')^3). \end{aligned} \tag{5.23} $$

Using (5.19) it is straightforward to show $P_6(\phi, \phi_0) = 0$. In fact, the first term in (5.23) vanishes for $\psi = \phi_0$ because of (5.18). The vanishing of the second term in (5.23) prompts us that the trilinear expression in ϕ is also an intertwining differential operator. This is indeed so, cf. the last section for some more examples for trilinear operators.

We can introduce now a hierarchy of $GL(2, \mathbb{R}$ invariant $\frac{1}{2}n$-differentials for every $n \in 2\mathbb{N}$:

$$\mathrm{Sch}_n(\phi) \doteq {}_2\mathcal{I}^0_{n\alpha}(\phi)\left(\frac{\mathrm{d}x}{\phi'}\right)^{n/2}, \quad n \in 2\mathbb{N}, \tag{5.24a}$$

$$\mathrm{Sch}_n(\phi \circ \phi_0) = \mathrm{Sch}_n(\phi) \circ \phi_0, \quad \phi_0(x) = \frac{\alpha x - \gamma}{\delta - \beta x} \tag{5.24b}$$

where property (5.24b) is just a restatement of (5.22) for $n > 2$ and (5.20) for $n = 2$. The usual Schwarzian Sch_4 is one of these objects (cf. (1.10)). It has an additional property:

$$\mathrm{Sch}_4(\phi \circ \psi) = \mathrm{Sch}_4(\phi) \circ \psi + \mathrm{Sch}_4(\psi), \quad \phi, \psi \in \mathrm{Diff}_0 S^1 \tag{5.25}$$

showing that it is a 1-cocycle on $\mathrm{Diff}_0 S^1$ [19,20].

Remark 5. One may consider also *half-differentials* and using (5.24a) and (5.16) write

$$\begin{aligned}\mathrm{Sch}_1(\phi) &\doteq {}_2\mathcal{I}^0_{\alpha}(\phi)\left(\frac{\mathrm{d}x}{\phi'}^{1/2}\right) \\ &= \phi(\phi'\,\mathrm{d}x)^{1/2} = \phi(\mathrm{d}\phi)^{1/2}.\end{aligned} \tag{5.26}$$

Property (5.24b) then follows from (5.20).

Finally, we just mention the case when the resulting functions are invariant: $c' = 0 \Longrightarrow c = -n$. This is only possible when n is even, cf. (3.3). Formula (5.12) with (3.3) substituted simplifies considerably

$$\begin{aligned}{}_2\mathcal{I}^{-n}_{n\alpha}(\phi) &= \sum_{j=0}^{n/2}(-1)^j\left(1 - \tfrac{1}{2}\delta_{j,n/2}\right)\phi^{(n-j)}\phi^{(j)} \\ &= \phi^n\phi - \phi^{(n-1)}\phi' + \phi^{n-2}\phi'' \\ &\quad - \phi^{(n-3)}\phi''' + \cdots + \tfrac{1}{2}(-1)^{n/2}(\phi^{(n/2)})^2, \quad n \in 2\mathbb{N}\end{aligned} \tag{5.27}$$

and we have fixed the constant γ_0 appropriately.

5.4

Now we consider the case $G = SL(3, \mathbb{R})$. We denote: $x = z^2_1$, $y = z^3_2$, $z = z^3_1$. The right action is ($\phi = \phi(x, y, z)$):

$$\hat{X}^-_1\phi = (\partial_x + y\partial_z)\phi, \quad \hat{X}^-_2\phi = \partial_y\phi, \quad \hat{X}^-_3\phi = \partial_z\phi, \tag{5.28}$$

and we have given it also for the non-simple root vector $X^-_3 = [X^-_2, X^-_1]$.

The bilinear operator corresponding to (3.5) is

$$
\begin{aligned}
{}^3_2\mathcal{I}^{\Lambda}_{\alpha}(\phi) = {} & \phi((\Lambda_1 - \Lambda_2)(\phi_{xy} + y\phi_{yz}) - \Lambda_2\phi_z) \\
& - (\Lambda_1 + \Lambda_2 + 1)\phi_y(\phi_x + y\phi_z),
\end{aligned} \tag{5.29}
$$

where $\alpha = \alpha_3 = \alpha_1 + \alpha_2$, $\phi_x \equiv \partial\phi/\partial_x$, etc. The two bilinear operators correponding to (3.6) are given by

$$
\begin{aligned}
{}^3_2\mathcal{I}^{\Lambda}_{2\alpha}(\phi) = {} & (a_1 + a_2 + 2a_3)\phi\phi_{zz} + (a_2 + 2a_3)\phi\phi_{xyz} \\
& + (a_2 + 4a_3)y \circ \phi\phi_{yzz} \\
& + a_3\phi(\phi_{xxyy} + y\phi_{xyyz} + y^2\phi_{yyzz}) \\
& + (b_1 + 2c_1)\phi_y(\phi_{xz} + y\phi_{zz}) \\
& + (b_2 + 2c_2)(\phi_x + y\phi_z)\phi_{yz} \\
& + c_1\phi_y(\phi_{xxy} + 2y\phi_{xyz} + y^2\phi_{yzz}) \\
& + c_2(\phi_x + y\phi_z)(\phi_{xyy} + y\phi_{yyz}) \\
& + (d_1 + d_2 + d_3)\phi_z^2 + (d_2 + 2d_3)\phi_z(\phi_{xy} + y\phi_{yz}) \\
& + d_3(\phi_{xy} + y\phi_{yz})^2 + d_4\phi_y^2(\phi_x + y\phi_z)^2
\end{aligned} \tag{5.30}
$$

with constants as given in (3.6b) and (3.6c).

The intertwining property is

$$
\begin{aligned}
& {}^3_2\mathcal{I}^{\Lambda}_{n\alpha} \circ (\tilde{\mathcal{L}}^{c}(X) \otimes 1_u + 1_u \otimes \tilde{\mathcal{L}}^{c}(X)) \\
& \quad = \tilde{\mathcal{L}}^{c'}(X) \circ {}^3_2\mathcal{I}^{\Lambda}_{n\alpha} \quad \forall X \in \mathcal{G},
\end{aligned} \tag{5.31}
$$

where $c_i = -\Lambda_i = -\Lambda(H_i)$, $\Lambda' = 2\Lambda - n\alpha$, $c'_i = -\Lambda'(H_i) = -2\Lambda(H_i) + n\alpha(H_i) = -2\Lambda_i + n = 2c_i + n$, $i = 1, 2$, since $\alpha(H_i) = (\alpha_1 + \alpha_2)(H_i) = 1$.

The case of invariant functions, i.e., $c_k = \epsilon_k = 0$ gives a trivial (zero) operator $n = 2$, while for $n = 1$ we have (up to scalar multiple):

$$
{}^3_2\mathcal{I}^{0}_{\alpha}(\phi) = \phi_y(\phi_x + y\phi_z). \tag{5.32}
$$

In the case of invariant resulting functions, i.e., $c'_k = \epsilon'_k = 0$, $\Lambda = \frac{1}{2}n\alpha$, $\Lambda_i = \frac{1}{2}n$, we have from (5.29):

$$
{}^3_2\mathcal{I}^{\alpha/2}_{\alpha}(\phi) = \tfrac{1}{2}\phi_z + 2\phi_y(\phi_z + y\phi_z), \tag{5.33}
$$

while from (5.30) we have two operators corresponding to the two solutions given by (3.6b) and (3.6c), respectively,

$$
\begin{aligned}
{}^3_2\mathcal{I}^{\alpha}_{2\alpha}(\phi) = {} & 2\phi\phi_{zz} - 2y\phi\phi_{yzz} \\
& - \phi(\phi_{xxyy} + y\phi_{xyyz} + y^2\phi_{yyzz}) \\
& - 2\phi_y(\phi_{xz} + y\phi_{zz}) + 6(\phi_x + y\phi_z)\phi_{yz} \\
& + 2\phi_y(\phi_{xxy} + 2y\phi_{xyz} + y^2\phi_{yzz}) \\
& + 2(\phi_x + y\phi_z)(\phi_{xyy} + y\phi_{yyz})
\end{aligned}
$$

$$-2\phi_z^2 - 2\phi_z(\phi_{xy} + y\phi_{yz}) - 2(\phi_{xy} + y\phi_{yz})^2, \tag{5.34a}$$

$$\tfrac{3}{2}\mathcal{I}'^{\alpha}_{2\alpha}(\phi) = \phi_y^2(\phi_x + y\phi_z)^2. \tag{5.34b}$$

6. Examples with $k \geq 3$

We return now to the $GL(2, \mathbb{R})$ setting to give examples of trilinear intertwining differential operators using the singular vectors of tri-Verma modules above. The trilinear intertwining differential operators for the case of $\mathcal{G} = sl(2, \mathbb{R})$ are given by the formula:

$$_3\mathcal{I}^{\Lambda}_{n\alpha}(\phi) = \sum_{\substack{j,k\in\mathbb{Z}+ \\ n-j-k\geq j\geq k}} \gamma^{\Lambda}_{njk}\phi^{(n-j-k)}\phi^{(j)}\phi^{(k)}, \tag{6.1}$$

where the coefficients γ^{Λ}_{nj} are given from the expressions for the corresponding singular vectors of tri- Verma modules, e.g., those given in Section 5.

If we pass to the group level then the possible weights are restricted to be 'integer' (as in Theorem 2): $\Lambda(H) = p \in Z$ and the corresponding intertwining property is

$$\begin{aligned} &_3\mathcal{I}^{\Lambda}_{n\alpha} \circ (T^{c,\epsilon}(g) \otimes T^{c,\epsilon}(g) \otimes T^{c,\epsilon}(g)) \\ &\quad = T^{c'\epsilon'}(g) \circ {}_3\mathcal{I}^{\Lambda}_{n\alpha} \quad \forall g \in G, \\ &\quad c = -\Lambda(H) = -p, \quad c' = 3(c+n) = 3(n-p), \\ &\quad \epsilon = \epsilon' = p (\mathrm{mod}\, 2). \end{aligned} \tag{6.2}$$

Next we restrict to the example of *invariant* functions ϕ, i.e., $c = \epsilon = 0$. The trilinear intertwining differential operators obtained from the singular vectors in (3.10) are:

$$_3\mathcal{I}^{0}_{\alpha}(\phi) = (\phi)^2\phi', \tag{6.3a}$$

$$_3\mathcal{I}^{0}_{2\alpha}(\phi) = \phi(\phi')^2, \tag{6.3b}$$

$$_3\mathcal{I}^{0}_{3\alpha}(\phi) = (\phi')^3, \tag{6.3c}$$

$$_3\mathcal{I}^{0}_{4\alpha}(\phi) = \phi(\phi'''\phi' - \tfrac{3}{2}(\phi'')^2), \tag{6.3d}$$

$$_3\mathcal{I}^{0}_{5\alpha}(\phi) = \phi'(\phi'''\phi' - \tfrac{3}{2}(\phi'')^2), \tag{6.3e}$$

$$_3\mathcal{I}^{0}_{6\alpha}(\phi) = \phi^{(4)}(\phi')^2 - 6\phi'''\phi''\phi' + 6(\phi'')^3, \tag{6.3f}$$

$$_3\mathcal{I}'^{0}_{6\alpha}(\phi) = \phi(\phi^{(5)}\phi' - 10\phi^{(4)}\phi'' + 10(\phi''')^2). \tag{6.3g}$$

We recall that the operator in (6.3f) has appeared in (5.23).

Analogously to Lemma 1 we note that for $n > 3$ the (formal) substitution $\phi(x) \mapsto (\alpha x - \gamma)/(\delta - \beta x)$ in the intertwining differential operators (6.3) gives zero:

$$_3\mathcal{I}^{0}_{n\alpha}(\phi_0) = 0, \quad \phi_0(x) \equiv \frac{\alpha x - \gamma}{\delta - \beta x}, \quad n > 3, \tag{6.4}$$

which because of the factorization follows from Lemma 1 except for (6.3f).

Analogously to Lemma 2 for $\phi, \psi \in \mathrm{Diff}_0 S^1$ one can check for the examples in (6.3):

$$_3\mathcal{I}^{0}_{n\alpha}(\phi \circ \psi) = (\psi')^n \, {}_3\mathcal{I}^{0}_{n\alpha}(\phi), \quad n = 1, 2, 3, \tag{6.5}$$

$$ {}_3\mathcal{I}^0_{5\alpha}(\phi\circ\psi) = (\psi')^5\, {}_3\mathcal{I}^0_{5\alpha}(\phi) + (\phi')^3\, {}_3\mathcal{I}^0_{5\alpha}(\psi), \tag{6.6a} $$

$$ {}_3\mathcal{I}^0_{6\alpha}(\phi\circ\psi) = (\psi')^6\, {}_3\mathcal{I}^0_{6\alpha}(\phi) + (\phi')^3\, {}_3\mathcal{I}^0_{6\alpha}(\psi) - 2\phi''(\phi'\psi')^2\, {}_2\mathcal{I}^0_{4\alpha}(\psi). \tag{6.6b} $$

Consider now the case of resulting invariant functions, i.e., $c' = \epsilon' = 0$, i.e., $p = \Lambda(H) = \hat{\Lambda} = n$. There is no operator for $n = 1$, while for $n > 1$ we get from (3.10):

$$ {}_3v_s^{2\alpha} = 2\phi''\phi^2 - \phi'^2\phi, \quad \hat{\Lambda} = 2, \tag{6.7a} $$

$$ {}_3v_s^{3\alpha} = 9\phi'''\phi^2 - 9\phi''\phi' + 4\phi'^3, \quad \hat{\Lambda} = 3, \tag{6.7b} $$

$$ {}_3v_s^{4\alpha} = 2\phi^{(4)}\phi^2 - 2\phi'''\phi'\phi + (\phi'')^2\phi, \quad \hat{\Lambda} = 4, \tag{6.7c} $$

$$ {}_3v_s^{5\alpha} = 25\phi^{(5)}\phi^2 - 25\phi^{(4)}\phi'\phi + 5\phi'''\phi''\phi + 16\phi'''\phi'^2 - 9(\phi'')^2\phi', \quad \hat{\Lambda} = 5, \tag{6.7d} $$

$$ {}_3v_5^{6\alpha} = 60\phi^{(4)}\phi''\phi - 50\phi^{(4)}\phi'^2 - 45\phi'''^2\phi + 60\phi'''\phi''\phi' - 24(\phi'')^3, \quad \hat{\Lambda} = 6, \tag{6.7e} $$

$$ {}_3v_s^{\prime 6\alpha} = 2\phi^{(6)}\phi^2 - 2\phi^{(5)}\phi'\phi + 2\phi^{(4)}\phi''\phi - \phi'''^2\phi, \quad \hat{\Lambda} = 6. \tag{6.7f} $$

We see that at lower levels there occur many factorizations and trilinear operators are actually determined by bilinear ones. We illustrate this by two statements for arbitrary k-Verma modules and the corresponding multilinear intertwining differential operators.

Proposition 5. *The singular vectors of the k-Verma modules ${}_kV^\Lambda$ of level $n\alpha$ with $n \in \mathbb{N}$, $n \le k$, $\alpha \in \Delta_S$, in the case $\Lambda(H_\alpha) = 0$ are given by:*

$$ {}_kv_s^{n\alpha} = \gamma_0 \left\{ \underbrace{X_\alpha^- \otimes \cdots \otimes X_\alpha^-}_{n} \otimes \underbrace{1_u \otimes \cdots \otimes 1_u}_{k-n} \right\} \hat{\otimes}\, v_0, \quad 1 \le n \le k \quad \Lambda(H_\alpha) = 0. \tag{6.8} $$

Proof. Follows from the direct verification. □

Proposition 6. *The $GL(2,\mathbb{R})$ multilinear intertwining differential operators with the property:*

$$ {}_k\mathcal{I}^0_{n\alpha} \circ \underbrace{T^{c,\epsilon}(g) \otimes \cdots \otimes T^{c,\epsilon}(g)}_{k} = T^{c',\epsilon'}(g) \circ {}_k\mathcal{I}^0_{n\alpha} \quad \forall g \in G, \quad c = -\Lambda(H) = 0, \quad c' = kn, \quad \epsilon = \epsilon' = 0 \tag{6.9} $$

are given by

$$ {}_k\mathcal{I}^0_{n\alpha}(\phi) = \phi^{n-k}(\phi')^n, \quad 1 \le n \le k \quad \Lambda(H) = 0. \tag{6.10} $$

Proof. Follows from Propositions 4 and 5. □

We note that the operators in (5.16), (5.17a) and (6.3a)–(6.3c) are partial cases of (6.10).

Appendix A. Tensor, symmetric and universal enveloping algebras

Let E be a vector space over F. The *tensor algebra* $T(E)$ over E is defined as the free algebra generated by the unit element. We have

$$T(E) = \bigoplus_{k=0}^{\infty} T_k(E),$$
$$T_k(E) \equiv \underbrace{E \otimes \cdots \otimes E}_{k}, \quad T_0(E) = F \cdot 1. \tag{A.1}$$

The elements $t \in T_k(E)$ are called covariant tensors of rank k:

$$t = \sum t^{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \tag{A.2}$$

where $e_i \in S$, S is a basis of E, $t^{i_s \cdots i_k} \in F$. (The rank of a covariant tensor does not depend on the choice of basis of E.) The tensor t is called symmetric tensor if $t^{i_n \cdots i_k}$ is symmetric in all indices. Let us denote

$$S(E) = \bigoplus_{k=0}^{\infty} S_k(E), \tag{A.3}$$

where $S_k(E)$ is the subspace of all symmetric tensor of rank k. Note that if $\dim E = n < \infty$, then

$$\dim\, S_k(E) = \binom{n+k-1}{k}. \tag{A.4}$$

Let $I(E)$, respectively, $I_k(E)$, be the two-sided ideal of $T(E)$, respectively, $T_k(E)$, generated by all elements of the type $x \otimes y - y \otimes x$, $x, y \in E$. Then we have

$$T(E) = S(E) \oplus I(E), \quad S(E) \cong T(E)/I(E),$$
$$S_k(E) \cong T_k(E)/I_k(E). \tag{A.5}$$

Consider now a Lie algebra $\mathcal{G}$. The universal enveloping algebra $U(\mathcal{G})$ of $\mathcal{G}$ is defined as the associative with generators e_i, where e_i forms a basis of $\mathcal{G}$ and the relations

$$e_i \otimes e_j - e_j \otimes e_i = \sum_k c_{ij}^k e_k \equiv [e_i, e_j] \tag{A.6}$$

hold, where c_{ij}^k are the structure constants of $\mathcal{G}$. Equivalently $U(\mathcal{G}) \cong T(\mathcal{G})/J(\mathcal{G})$, where $T(\mathcal{G})$ is the tensor algebra over $\mathcal{G}$, $J(\mathcal{G})$ is the ideal generated by the elements $[x, y] - (x \otimes y - y \otimes x)$. Since $T(\mathcal{G})U(\mathcal{G}) \oplus J(\mathcal{G}) = S(\mathcal{G}) \oplus I(\mathcal{G})$ and $J(\mathcal{G}) \cong I(\mathcal{G})$ are isomorphic, then also $U(\mathcal{G}) \cong S(\mathcal{G})$ as vector spaces. This is the content of the Poincaré–Birkhoff–Witt (PBW) theorem.

Further in the case of $U(\mathcal{G})$ we shall omit the $\otimes$ signs in the expressions for its elements. With this convention as a consequence of the PBW theorem $U(\mathcal{G})$ has the following basis:

$$e_0 = 1, \qquad e_{i_1 \cdots i_k} = e_{i_1} \cdots e_{i_k}, \quad i_1 \leq \cdots \leq i_k, \tag{A.7}$$

where we are assuming some ordering of the basis of $\mathcal{G}$, e.g., the lexicographic one.

Finally we recall that $\mathcal{G}$ and $U(\mathcal{G})$ are completely determined from the following commutation relations:

$$[H_i, H_j] = 0, \qquad [H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm}, \tag{A.8a}$$

$$[X_i^+, X_j^-] = \delta_{ij} H_i \tag{A.8b}$$

and Serre relations

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{n-k} = 0,$$
$$i \neq j, \quad n = 1 - a_{ij}, \tag{A.9}$$

where $X_i^{\pm}$, H_i, $i = 1, \ldots, l = \text{rank}\ \mathcal{G}$ are the Chevalley generators of $\mathcal{G}$ (corresponding to the simple roots α_i), $(a_{ij}) = (2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i))$ is the Cartan matrix of $\mathcal{G}$, and $(\cdot\ ,\ \cdot)$ is normalized so that for the short roots α we have $(\alpha, \alpha) = 2$.

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